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BMN correlators by loop equations

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Abstract: In the BMN approach to $\mathcal{N}=4$ SYM a large class of correlators of interest are expressible in terms of expectation values of traces of words in a zero-dimensional gaussian complex matrix model. We develop a loop-equation based, analytic strategy for evaluating such expectation values to any order in the genus expansion. We reproduce the expectation values which were needed for the calculation of the one-loop, genus one correction to the anomalous dimension of BMN-operators and which were earlier obtained by combinatorial means. Furthermore, we present the expectation values needed for the calculation of the one-loop, genus two correction.

Keywords: 1/N Expansion, Penrose limit and pp-wave background, Matrix Models.

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Contents

1. Introduction

Recent progress in string and gauge theory [1]–[3] has brought to light an interesting pp-wave/BMN-correspondence which is a special version of the celebrated AdS/CFT correspondence. The pp-wave is a ten-dimensional geometry which can be obtained as a Penrose limit of $AdS_5 \times S_5$ and which constitutes a background where it is possible to quantize type-IIB string theory in light cone gauge [1, 2, 4]. BMN stands for Berenstein, Maldacena and Nastase who identified the gauge theory dual as a special sector of $\mathcal{N} = 4$ SYM based on gauge group SU(N) with a certain limit understood.

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The BMN sector of $\mathcal{N}=4$ SYM consists of operators which carry a large R-charge, J, associated with a selected SO(2) sub-group of the full SO(6) R-symmetry group and for which $\Delta - J$ is finite where Δ is the conformal dimension. It has been argued that the quantum corrections to correlation functions involving such operators only depend on g_{YM} via the parameter $\lambda' = (g_{YM}^2 N)/J^2$ [3, 5, 6] and the BMN limit is given by the scaling prescription

$$g_{YM} \quad \text{fixed}\,, \quad J, N \to \infty \qquad \text{with } g_2 = \frac{J^2}{N} \text{ fixed} \tag{1.1}$$

which in particular renders λ' finite.[1] As shown in [7, 8, 9] despite being a large-N limit the BMN limit is not a planar limit. Diagrams of all genera survive the limit and contributions of genus h are weighted by a factor $(g_2)^{2h}$. One could say that the BMN approach to $\mathcal{N}=4$ SYM introduces a new 't Hooft expansion with a new gauge coupling constant λ' and a new genus counting parameter, g_2. However, the BMN *limit* is not a new 't Hooft *limit* because the genus counting parameter remains finite as the limit is taken. Rather, the BMN limit is an interesting new double scaling limit much like the one encountered in the study of 2D quantum gravity [10, 11, 12].

To introduce the R-charge of the BMN approach we single out two of the six scalars $\phi_i(x)$, $i = \{1, \ldots, 6\}$, which transform under the SO(6) R-symmetry group, say ϕ_5 and ϕ_6, and form the complex combination

$$Z(x) = \frac{1}{\sqrt{2}}\left(\phi_5(x) + i\phi_6(x)\right)\,. \tag{1.2}$$

Then we define the R-charge, J, as the quantum number conjugate to the phase of Z. As mentioned above, operators which survive the BMN limit are characterized by having J very large and $\Delta - J$ finite. In practice this means that such operators contain a large number of Z-fields and a finite number of impurities in the form of fields not carrying R-charge such as ϕ_1, ϕ_2, ϕ_3 and ϕ_4. In $\mathcal{N}=4$ SYM and in particular in its BMN sector the space-time dependence of two- and three-point functions is fixed by conformal invariance. At the classical level the calculation of such correlators then reduces to the calculation of expectation values in a zero dimensional gaussian complex matrix model. For protected operators this statement trivially remains true when interactions are included and for non-protected operators a similar simplification can be obtained even at the quantum level if one introduces effective vertices [7, 9]. (This procedure has so far only been implemented at one-loop level.) In reference [13] it was proposed that only two-point functions of appropriately defined multi-trace operators would have a string theory interpretation and this point of view has been supported by gauge theory calculations [14, 15]. This implies that extracting information about pp-wave strings from the gauge-theory reduces to determining the expectation value of traces of words in a zero-dimensional gaussian complex matrix model. So far genuine matrix model techniques have only been exploited in the calculation of a very limited set of expectation values [7, 9] whereas the major part of those obtained were determined by combinatorial means. For higher genera combinatorial arguments become very involved. From the string theory point of view higher genera contributions are

[1] It appears that this limit is the same for gauge groups SU(N) and U(N). In this paper we shall be considering gauge group U(N).

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most interesting because they encode information about string interactions. So far gauge theory calculations were only pursued up to and including genus one.

In the present paper we shall develop a loop-equation based, analytic strategy which allows us to calculate by recursion expectation values of products of arbitrary traces of words in a gaussian complex one-matrix model to any order in the genus expansion. The outline of our paper is the following. First, in section 2 we explain in more detail how to reduce the calculation of two-point functions in $\mathcal{N} = 4$ SYM to the calculation of matrix model expectation values, focusing on the two-point function of the so-called BMN operators. Next, in section 3 we introduce the notation necessary for our matrix model investigations and list the matrix model expectation values which are needed to find respectively the genus one and the genus two, one-loop correction to the anomalous dimension of the BMN-operators. In section 4 we derive the two basic relations on which all our considerations are based; the split and merge rule respectively. As a first application of these rules we reproduce in section 5 all the matrix model expectation values needed for the above mentioned genus one calculation by purely analytic computations. Subsequently, in section 6 we determine the correlators needed for the genus 2 calculation and finally in section 7 we show how our strategy allows us to find the expectation value of traces of arbitrary words to any order in the genus expansion. Section 8 is devoted to correlators which can be calculated exactly and section 9 contains our conclusions.

Note: as we were completing our manuscript a related, interesting paper appeared where another loop equation based technique is applied to the study of the (planar) BMN-limit [16].

2. From $\mathcal{N} = 4$ SYM to matrix model

The field content of $\mathcal{N} = 4$ SYM in four dimensions consists of the scalars $\phi_i(x)$, $i \in \{1, \ldots, 6\}$, a space-time vector $A_\mu(x)$ and a sixteen component spinor $\psi(x)$. These fields are hermitean $N \times N$ matrices and can be expanded in terms of the generators T^a of the gauge group $\mathrm{U}(N)$, for instance

$$(\phi_i)_{\alpha\beta}(x) = \sum_{a=0}^{N^2-1} \phi_i^a(x) T^a_{\alpha\beta} \,. \tag{2.1}$$

The generators are normalized as follows

$$\mathrm{tr}[T^a, T^b] = \delta^{ab} \,, \qquad \sum_{a=0}^{N^2-1} T^a_{\alpha\beta} T^a_{\gamma\delta} = \delta_{\delta\alpha}\delta_{\beta\gamma} \tag{2.2}$$

and the euclidean action reads

$$\begin{aligned} S = \frac{2}{g_{YM}^2} \int d^4x \,\mathrm{tr}\bigg(& \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \phi_i D_\mu \phi_i - \frac{1}{4} [\phi_i, \phi_j][\phi_i, \phi_j] + \\ & + \frac{1}{2} \bar{\psi} \Gamma_\mu D_\mu \psi - \frac{i}{2} \bar{\psi}\, \Gamma_i [\phi_i, \psi] \bigg) \,, \end{aligned} \tag{2.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and the covariant derivative is $D_\mu \phi_i = \partial_\mu \phi_i - i[A_\mu, \phi_i]$. Furthermore, (Γ_μ, Γ_i) are the ten-dimensional Dirac matrices in the Majorana-Weyl representation. Working in Feynman gauge, the propagators of the scalar fields take the form

$$\langle (\phi_i)_{\alpha\beta}(x)(\phi_j)_{\gamma\delta}(0)\rangle = \frac{g_{YM}^2}{8\pi^2 x^2}\,\delta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{2.4}$$

and in particular (cf. eqn. (1.2))

$$\langle \bar{Z}_{\alpha\beta}(x) Z_{\gamma\delta}(0)\rangle = \frac{g_{YM}^2}{8\pi^2 x^2}\,\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{2.5}$$

$$\langle \bar{Z}_{\alpha\beta}(x) \bar{Z}_{\gamma\delta}(0)\rangle = \langle Z_{\alpha\beta}(x) Z_{\gamma\delta}(0)\rangle = 0\,. \tag{2.6}$$

Operators $\mathcal{O}(x)$ which belong to the BMN sector of $\mathcal{N}=4$ SYM are characterized by containing a large number of Z-fields and a finite number of impurities in the form of fields not carrying R-charge. As an example, let us consider the most studied, so-called BMN-operator

$$\mathcal{O}^J_{12,n}(x) \equiv \frac{1}{\sqrt{N^{J+2}J}} \sum_{p=0}^{J} e^{2\pi i pn/J}\, \mathrm{tr}\left(\phi_1(x) Z^p(x) \phi_2(x) Z^{J-p}(x)\right)\,. \tag{2.7}$$

From the Feynman rules (2.4), (2.5) and (2.6), (or alternatively from conformal invariance) it follows that the tree level two-point function of BMN-operators can be written as

$$\langle \mathcal{O}^J_{12,n}(x)\bar{\mathcal{O}}^J_{12,m}(0)\rangle = \left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^{J+2} \sum_{p,q=0}^{J} e^{2\pi i(np-mq)/J} \langle \mathrm{tr}(\phi_1 Z^p \phi_2 Z^{J-p})\,\mathrm{tr}(\phi_1 \bar{Z}^{J-q}\phi_2 \bar{Z}^q)\rangle\,,$$

where the space-time independent matrix valued fields, ϕ_i and Z should be contracted using the following Feynman rules

$$\langle (\phi_i)_{\alpha\beta}(\phi_j)_{\gamma\delta}\rangle = \delta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{2.8}$$

$$\langle \bar{Z}_{\alpha\beta} Z_{\gamma\delta}\rangle = \delta_{\alpha\delta}\delta_{\beta\gamma}\,. \tag{2.9}$$

The contraction of the ϕ-fields can easily be done by hand and we are left with

$$\langle \mathcal{O}^J_{12,n}(x)\bar{\mathcal{O}}^J_{12,m}(0)\rangle = \left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^{J+2} \sum_{p,q=0}^{J} e^{2\pi i(np-mq)/J} \langle \mathrm{tr}(Z^{J-p}\bar{Z}^{J-q})\,\mathrm{tr}(Z^p \bar{Z}^q)\rangle\,. \tag{2.10}$$

Now, the remaining expectation value can be identified as an expectation value in a zero dimensional gaussian complex matrix model, namely

$$\langle \mathrm{tr}(Z^{J-p}\bar{Z}^{J-q})\,\mathrm{tr}(Z^p\bar{Z}^q)\rangle = \int dZ d\bar{Z} \exp\left(-\,\mathrm{tr}(\bar{Z}Z)\right) \mathrm{tr}(Z^{J-p}\bar{Z}^{J-q})\,\mathrm{tr}(Z^p\bar{Z}^q)\,. \tag{2.11}$$

Here $dZd\bar{Z}$ is defined as

$$dZd\bar{Z} = \prod_{i,j=1}^{N} \frac{d\,\mathrm{Re}\,Z_{ij}\,d\,\mathrm{Im}\,Z_{ij}}{\pi} \tag{2.12}$$

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such that $\int dZd\bar{Z}e^{-\operatorname{tr}(\bar{Z}Z)} = 1$. The identification (2.11) holds because the matrix model measure (2.12) combined with the gaussian action precisely give rise to the contraction rule (2.9). The action and the measure carry a $\mathrm{U}(N) \times \mathrm{U}(N)$ symmetry corresponding to the transformation $Z \to UZV^\dagger$ with U and V unitary. Expectation values of operators likewise carrying this symmetry, i.e. traces of products of $(\bar{Z}Z)$ can be calculated even for arbitrary $\mathrm{U}(N)\times\mathrm{U}(N)$ invariant potential order by order in the genus expansion using loop equations [17]. Expectation values of operators consisting of products of traces involving only Z's or $\bar{Z}$'s can likewise be obtained by well-established methods, namely by character expansion [18, 19] or by the method of Ginibre [20]. Notice that the object appearing in (2.11) does not belong to either of these classes of correlators. The aim of the present paper is to develop a method which allows us to deal with general correlators composed of traces of arbitrary words of Z and $\bar{Z}$.

It is obvious that any tree-level two point function of operators in the BMN sector of $\mathcal{N} = 4$ SYM can be reduced in the above manner. One pulls out the space-time factor, contract by hand the *finite* number of impurities and one is left with a matrix model expectation value. By making use of so-called effective vertices one can also reduce one-loop corrections to two-point functions to matrix model expectation values [7, 9].

As explained in the introduction correlation functions in the BMN sector of $\mathcal{N} = 4$ SYM have an expansion in powers of J^4/N^2, the genus counting parameter, and we are interested in determining (at least) the first terms in this expansion. For that purpose it is convenient to decompose our matrix model expectation values into connected and disconnected parts, f. inst.

$$\langle \operatorname{tr}(Z^p\bar{Z}^q)\operatorname{tr}(Z^{J-p}\bar{Z}^{J-q})\rangle = \langle \operatorname{tr}(Z^p\bar{Z}^q)\rangle\langle \operatorname{tr}(Z^{J-p}\bar{Z}^{J-q})\rangle + \langle \operatorname{tr}(Z^p\bar{Z}^q)\operatorname{tr}(Z^{J-p}\bar{Z}^{J-q})\rangle_{\mathrm{conn}}$$

as the connected part is down by a factor of $1/N^2$ compared to the disconnected one.

Furthermore, it is convenient to work with generating functionals for expectation values in stead of working with the expectation values themselves. For instance, let us define

$$W_{1,1}(x_1, y_1; x_2, y_2) = \left\langle \operatorname{tr}\left(\frac{1}{x_1 - Z}\frac{1}{\bar{y}_1 - \bar{Z}}\right)\operatorname{tr}\left(\frac{1}{x_2 - Z}\frac{1}{\bar{y}_2 - \bar{Z}}\right)\right\rangle_{\mathrm{conn}}, \tag{2.13}$$

where x_1, y_1, x_2, y_2 are to be viewed as auxiliary variables. Then we have

$$\begin{aligned} &W_{1,1}\left(X\mathrm{e}^{(-i\pi n)/J}, X\mathrm{e}^{(-i\pi m)/J}; X\mathrm{e}^{(i\pi n)/J}, X\mathrm{e}^{(i\pi m)/J}\right) = \\ &\quad = e^{i\pi(m-n)}\sum_{J=0}^{\infty}(X\bar{X})^{-J-2}\sum_{p,q=0}^{J}\left\langle \operatorname{tr}(Z^{J-p}\bar{Z}^{J-q})\operatorname{tr}(Z^p\bar{Z}^q)\right\rangle_{\mathrm{conn}}\mathrm{e}^{2i\pi(np-mq)/J} \end{aligned} \tag{2.14}$$

which immediately allows us to extract the sum appearing in (2.10) also known as the tree-level mixing matrix. So far, the tree-level mixing matrix has been calculated to order J^4/N^2 (genus one) in [7, 9] and to order $(J^4/N^2)^2$ in [9]. Furthermore, the one-loop correction to the two-point function was calculated to genus one in [7, 9]. In the following section we list the matrix model expectation values or rather the generating functions needed for that computation. We likewise list the ones needed to extend that calculation to genus two. Later we shall determine all of these functions..

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3. Definitions and notation

We consider a complex gaussian matrix model whose partition function is given by

$$\mathcal{Z} = \int d\mu \, \mathrm{e}^{-S} = \int \mathrm{d}Z \mathrm{d}\bar{Z} \, \mathrm{e}^{-N \,\mathrm{tr}\, Z\bar{Z}} , \tag{3.1}$$

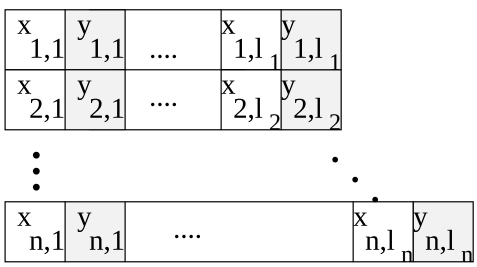
where the integration runs over complex $N \times N$ matrices. Note that there appears a factor of N in front of the action. This factor is introduced only for convenience and can easily be scaled away in the final results. Let us define the following generating functionals, also denoted as loop functions.

$$\begin{aligned}
\omega(x) &= \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{x-Z} \right\rangle = \frac{1}{x}, \qquad \bar{\omega}(y) = \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{\bar{y}-\bar{Z}} \right\rangle = \frac{1}{\bar{y}} && (3.2)\\
W_1(x,y) &= \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{x-Z}\frac{1}{\bar{y}-\bar{Z}} \right\rangle && (3.3)\\
W_2(x,y,x',y') &= \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{x-Z}\frac{1}{\bar{y}-\bar{Z}}\frac{1}{x'-Z}\frac{1}{\bar{y}'-\bar{Z}} \right\rangle && (3.4)\\
W_{1,1}(x,y;x',y') &= \left\langle \mathrm{tr}\, \frac{1}{x-Z}\frac{1}{\bar{y}-\bar{Z}} \,\mathrm{tr}\, \frac{1}{x'-Z}\frac{1}{\bar{y}'-\bar{Z}} \right\rangle_{\mathrm{conn}} && (3.5)\\
U_1(x;x',y') &= \left\langle \mathrm{tr}\, \frac{1}{x-Z} \,\mathrm{tr}\, \frac{1}{x'-Z}\frac{1}{\bar{y}'-\bar{Z}} \right\rangle_{\mathrm{conn}} . && (3.6)
\end{aligned}$$

We have normalized these functions so that their leading term in the large-N expansion is of order one. Knowing the leading order contributions to these functions for large N as well as the next to leading order contribution to $W_1(x,y)$ suffices for the calculation of the one-loop, genus one correction to the anomalous dimension of the BMN operators. However, we shall be interested in more general loop functions. We define

$$\begin{aligned}
&W_{l_1,\ldots,l_n}(x_{1,1},y_{1,1},\ldots,x_{1,l_1},y_{1,l_1};\ldots;x_{n,1},y_{n,1},\ldots,x_{n,l_n},y_{n,l_n}) = \\
&\quad = N^{n-2}\left\langle \prod_{j=1}^{n} \mathrm{tr}\left(\prod_{i=1}^{l_j} \frac{1}{x_{j,i}-Z}\frac{1}{\bar{y}_{j,i}-\bar{Z}} \right) \right\rangle_{\mathrm{conn}} \\
&\quad = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W^{(h)}_{l_1,\ldots,l_n}(x_{1,1},y_{1,1},\ldots,x_{1,l_1},y_{1,l_1};\ldots;x_{n,1},y_{n,1},\ldots,x_{n,l_n},y_{n,l_n}) . \qquad (3.7)
\end{aligned}$$

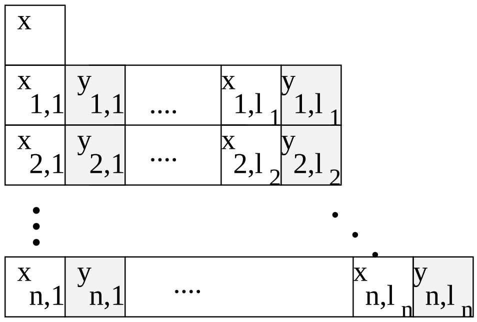
This function is invariant under permutation of the various traces, under cyclic permutation of the factors inside a given trace and it is changed to its complex conjugate under $x \leftrightarrow y$. We can represent it with a Young diagram like graph as follows with $l_1 \leq l_2 \leq \cdots \leq l_n$



We also define

$$
\begin{aligned}
&U_{l_1,\ldots,l_n}(x;x_{1,1},y_{1,1},\ldots,x_{1,l_1},y_{1,l_1};\ldots;x_{n,1},y_{n,1},\ldots,x_{n,l_n},y_{n,l_n}) = \\
&\quad = N^{n-1}\left\langle \mathrm{tr}\,\frac{1}{x-Z}\prod_{j=1}^{n}\mathrm{tr}\left(\prod_{i=1}^{l_j}\frac{1}{x_{j,i}-Z}\frac{1}{\bar{y}_{j,i}-\bar{Z}}\right)\right\rangle_{\mathrm{conn}} \\
&\quad = \sum_{h=0}^{\infty}\frac{1}{N^{2h}}U^{(h)}_{l_1,\ldots,l_n}(x;x_{1,1},y_{1,1},\ldots,x_{1,l_1},y_{1,l_1};\ldots;x_{n,1},y_{n,1},\ldots,x_{n,l_n},y_{n,l_n}) \qquad (3.8)
\end{aligned}
$$

which we can similarly represent with a Young diagram like graph



where $l_1 \leq l_2 \leq \cdots \leq l_n$. Calculating the one-loop, *genus two* correction to the two-point function of BMN-operators would require the knowledge of the third order contribution to $W_1(x,y)$, the next to leading order contribution to the functions (3.4)–(3.6) as well as the leading order contribution to the functions $U_2(x;x_1,y_1,x_2,y_2)$, $W_3(x_1,y_1,x_2,y_2,x_3,y_3)$, $U_{1,1}(x;x_1,y_1;x_1',y_1')$ and $W_{1,2}(x,y;x_1,y_1,x_2,y_2)$. In section 6 we shall show how to determine these and in section 7 we shall describe a general strategy for determining any multi-loop function or equivalently any expectation value of traces of words to any order in the genus expansion.

4. Loop equations

All our computations will be based on two simple rules which can be derived by loop equation techniques, based on the fact that the matrix model partition function is invariant under field redefinitions [22]. Here we restrict ourselves to considering the case of a complex matrix model with a gaussian potential which is the case of interest for the BMN sector of $\mathcal{N}=4$ SYM. This is, however, just an almost trivial application of a method which works under much more general circumstances and will be presented for a hermitean two-matrix model with arbitrary U(N) invariant potentials in [23]. (The hermitian two-matrix model with cubic potential was studied by loop equation methods in reference [24].)

4.1 Split rule

Consider the field redefinition

$$
Z \to Z + \epsilon A\frac{1}{x-Z}B\,. \qquad (4.1)
$$

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This redefinition gives rise to the following change of the measure

$$\delta\left(dZd\bar{Z}\right)=2\,\mathrm{Re}\left(\epsilon\;\mathrm{tr}\,A\frac{1}{x-Z}\,\mathrm{tr}\,\frac{1}{x-Z}B\right)dZd\bar{Z}+O(\epsilon^2)\,. \tag{4.2}$$

If A or B depends on Z or $\bar{Z}$ there will be additional contributions which are obtained by applying the usual chain rule in combination with the split and merge rules. Obviously, under (4.1) the action changes as

$$\delta S=2N\,\mathrm{Re}\left(\epsilon\;\mathrm{tr}\left(\bar{Z}A\frac{1}{x-Z}B\right)\right)\,. \tag{4.3}$$

The relations (4.2) and (4.3) hold for arbitrary complex ϵ, in particular for ϵ purely real or purely imaginary. Therefore we conclude

$$\left\langle\mathrm{tr}\,A\frac{1}{x-Z}\,\mathrm{tr}\,\frac{1}{x-Z}B\right\rangle=N\left\langle\mathrm{tr}\,\bar{Z}A\frac{1}{x-Z}B\right\rangle \tag{4.4}$$

4.2 Merge rule

Here we consider the following field redefinition

$$Z\to Z+\epsilon A\,\mathrm{tr}\,\frac{1}{x-Z}B \tag{4.5}$$

for which the change in the measure is

$$\delta\left(dZd\bar{Z}\right)=2\,\mathrm{Re}\left(\epsilon\,\mathrm{tr}\,A\frac{1}{x-Z}B\frac{1}{x-Z}\right)dZd\bar{Z}+O(\epsilon^2)\,. \tag{4.6}$$

Again, if A or B depends on Z or $\bar{Z}$ there will be additional contributions which are obtained by applying the usual chain rule in combination with the split and merge rules. The change of the action is obvious and our final merge rule reads

$$\left\langle\mathrm{tr}\left(A\frac{1}{x-Z}B\frac{1}{x-Z}\right)\right\rangle=N\left\langle\mathrm{tr}\,\bar{Z}A\;\;\mathrm{tr}\,\frac{1}{x-Z}B\right\rangle\,. \tag{4.7}$$

5. Functionals needed for the one-loop, genus one computation

In this section we shall determine the leading order contribution for large N to the loop-functions (3.3)–(3.6) as well as the next to leading order contribution to (3.3). As mentioned above these are the objects needed for the computation of the one-loop, genus one correction to the anomalous dimension of the BMN-operators. In section 8 we will show that the functionals (3.3) and (3.6) can be calculated exactly but to expose the completeness of our loop equation method we shall derive the leading order contributions to these below as well.

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5.1 $\omega(x)$ and $W_1(x,y)$ to leading order

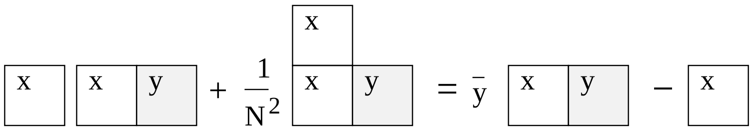
Considering the field redefinition $\delta\bar{Z} = 1/(x - Z)$ we easily get

$$0 = -1 + x\omega(x) \tag{5.1}$$

which is true to all orders in $1/N^2$ and gives

$$\omega(x) = \frac{1}{x}\,. \tag{5.2}$$

This result of course trivially follows from symmetry arguments. Next, we make use of the field redefinition $\delta Z = (1/(x - Z))(1/(\bar{y} - \bar{Z}))$ and obtain



$$\omega(x)W_1(x,y) + \frac{1}{N^2}U_1(x;x,y) = \bar{y}W_1(x,y) - \omega(x)\,. \tag{5.3}$$

Above, a space between two Young diagrams signifies multiplication of the corresponding functions. To leading order in $\frac{1}{N^2}$ we can neglect the second term in (5.3) and we get

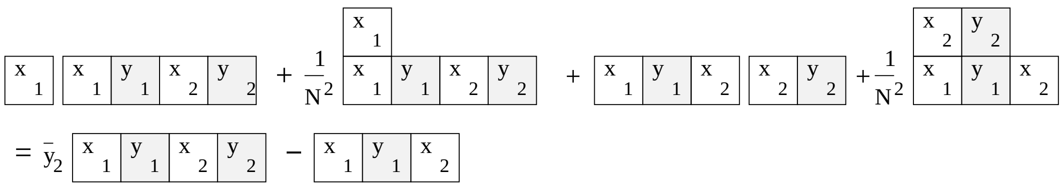
$$W_1^{(0)}(x,y) = \frac{1}{x\bar{y} - 1} \tag{5.4}$$

which is in accordance with the simple combinatorial result

$$\frac{1}{N}\left\langle \mathrm{tr}\, Z^J \bar{Z}^J \right\rangle = 1 + O\left(\frac{1}{N^2}\right). \tag{5.5}$$

5.2 W_2, U_1 and $W_{1,1}$ to leading order

Performing the change of variable $\delta Z = (1/(x_1 - Z))(1/(\bar{y}_1 - \bar{Z}))(1/(x_2 - Z))(1/(\bar{y}_2 - \bar{Z}))$ leads to:



$$\begin{aligned} W_2(x_1,y_1,x_2,y_2)\left(\bar{y}_2 - \frac{1}{x_1}\right) &= (1 + W_1(x_2,y_2))\,\frac{W_1(x_1,y_1) - W_1(x_2,y_1)}{x_2 - x_1} + \\ &\quad + \frac{1}{N^2}U_2(x_1;x_1,y_1,x_2,y_2) + \\ &\quad + \frac{1}{N^2}\,\frac{W_{1,1}(x_1,y_1;x_2,y_2) - W_{1,1}(x_2,y_1;x_2,y_2)}{x_2 - x_1}\,. \end{aligned} \tag{5.6}$$

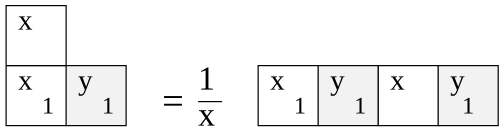
In the equation above we have used fractional decomposition to express the quantities represented by the two last Young diagrams in each line in terms of usual W functions with an even number of arguments. From equation (5.6) we can easily find the leading contribution to $W_2(x_1, y_1, x_2, y_2)$ for large N, namely

$$W_2^{(0)}(x_1, y_1, x_2, y_2) = \frac{x_1 \bar{y}_1 x_2 \bar{y}_2}{(x_1\bar{y}_1 - 1)(x_2\bar{y}_1 - 1)(x_1\bar{y}_2 - 1)(x_2\bar{y}_2 - 1)} \tag{5.7}$$

which reproduces the combinatorial result

$$\frac{1}{N}\left\langle \operatorname{tr} Z^{J-p}\bar{Z}^{J-q}Z^p\bar{Z}^q\right\rangle = 1 + \operatorname{Min}\left[p, q, J-p, J-q\right] + O\left(\frac{1}{N^2}\right). \tag{5.8}$$

Next, carrying out the change of variables $\delta\bar{Z} = (1/(x - Z))\operatorname{tr}(1/((x_1 - Z)(\bar{y}_1 - \bar{Z})))$ we get the following simple relation



$$U_1(x; x_1, y_1) = \frac{1}{x} W_2(x_1, y_1, x, y_1) \tag{5.9}$$

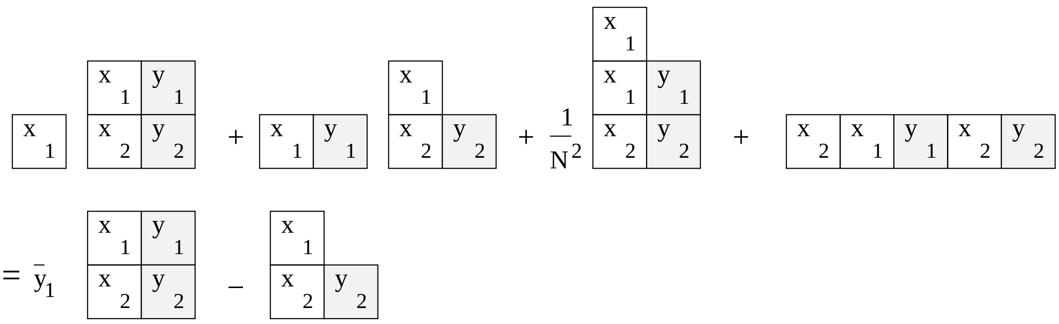
i.e. to leading order

$$U_1^{(0)}(x; x_1, y_1) = \frac{x_1\bar{y}_1^2}{(x_1\bar{y}_1 - 1)^2(x\bar{y}_1 - 1)^2} \tag{5.10}$$

which is in agreement with the combinatorial result

$$\left\langle \operatorname{tr}\bar{Z}^p \operatorname{tr} Z^J \bar{Z}^{J-p}\right\rangle_{\text{conn}} = p(J - p + 1) + O\left(\frac{1}{N^2}\right). \tag{5.11}$$

Finally, we consider the field redefinition $\delta Z = (1/(x_1 - Z))(1/(\bar{y}_1 - \bar{Z}))\operatorname{tr}(1/((x_2 - Z) \times (\bar{y}_2 - \bar{Z})))$ which leads to the following equation for $W_{1,1}(x_1, y_1; x_2, y_2)$



$$\begin{aligned} W_{1,1}(x_1, y_1; x_2, y_2)\left(\bar{y}_1 - \frac{1}{x_1}\right) = &\ (1 + W_1(x_1, y_1))\, U_1(x_1; x_2, y_2) + \\ &+ \frac{W_2(x_1, y_1, x_2, y_2) - W_2(x_2, y_1, x_2, y_2)}{x_2 - x_1} + \\ &+ \frac{1}{N^2} U_{1,1}(x_1; x_1, y_1; x_2, y_2)\,. \end{aligned} \tag{5.12}$$

Now, making use of (5.4), (5.7) and (5.10) we get

$$W^{(0)}_{1,1}(x_1, y_1; x_2, y_2) = \frac{x_1 x_2 \bar{y}_1 \bar{y}_2 (1 + x_1 x_2 \bar{y}_1 \bar{y}_2 (x_1 \bar{y}_2 + x_2 \bar{y}_1 - 3))}{(x_1\bar{y}_1 - 1)^2 (x_1\bar{y}_2 - 1)^2 (x_2\bar{y}_1 - 1)^2 (x_2\bar{y}_2 - 1)^2}\,, \tag{5.13}$$

where we note that

$$(1 + x_1 x_2 \bar{y}_1 \bar{y}_2 (x_1 \bar{y}_2 + x_2 \bar{y}_1 - 3)) = \det \begin{pmatrix} 1 & x_1\bar{y}_2 & x_1\bar{y}_2 \\ x_2\bar{y}_1 & 1 & x_2\bar{y}_1 \\ x_2\bar{y}_1 & x_1\bar{y}_2 & 1 \end{pmatrix}. \tag{5.14}$$

From (5.13) we can easily get the genus one correction to the tree-level mixing matrix of BMN operators coming from connected diagrams, namely (cf. eqn (2.14))

$$\begin{aligned} C^{(0)}_{n,m} &= \sum_{p,q=0}^{J} \left\langle \mathrm{tr}(Z^{J-p}\bar{Z}^{J-q})\,\mathrm{tr}(Z^p\bar{Z}^q)\right\rangle_{\mathrm{conn}} e^{2\pi i(np-mq)/J} \\ &= \oint \frac{dr}{2\pi i}\, r^{J+1} W_{1,1}(\sqrt{r}e^{-i\pi n/J}, \sqrt{r}e^{-i\pi m/J}; \sqrt{r}e^{i\pi n/J}, \sqrt{r}e^{i\pi m/J})\, e^{i\pi(n-m)} \\ &= \oint \frac{dr}{2\pi i} \frac{r^{J+3}(1 + r^2[2r\cos(((n+m)\pi)/J) - 3])\, e^{i\pi(n-m)}}{(r - e^{-i\pi(n-m)/J})^2 (r - e^{-i\pi(n+m)/J})^2 (r - e^{i\pi(n-m)/J})^2 (r - e^{i\pi(n+m)/J})^2}\,. \end{aligned} \tag{5.15}$$

It is obvious that the analyticity structure of the integrand depends on the values of n and m, more precisely we have

- $n = m = 0$: A pole of order 8 at $r = 1$.
- $n = 0$ and $m \neq 0$ (or $m = 0$ and $n \neq 0$): Two poles of order 4 at $r = e^{\pm i\pi n/J}$ (or at $r = e^{\pm i\pi m/J}$).
- $n = m \neq 0$ or $n = -m \neq 0$: One pole of order 4 at $r = 1$ and 2 poles of order 2 at $r = e^{\pm 2i\pi n/J}$ Notice that the residues are not the same in the two cases.
- $|n| \neq |m|$ and $n \neq 0 \neq m$: Four poles of order two at $r = e^{\pm i\pi(n-m)/J}$ and $r = e^{\pm i\pi(n+m)/J}$.

Strictly speaking, the conditions on n and m are to be understood modulo J but we always consider n, $m \ll J$. The fact that the evaluation of the mixing matrix has to be split into 5 separate cases follows immediately in the generating functional picture and evaluating the contour integral (5.15) we easily reproduce the result of reference [7] i.e.

$$C^{(0)}_{n,m} = C^{(0)}_{m,n} = J^5 \begin{cases} \frac{1}{40} & n = m = 0 \\ \frac{3 - 2\pi^2 n^2}{24\, n^4 \pi^4} & n \neq 0, m = 0 \\ \frac{21 - 2\pi^2 n^2}{48\, n^4 \pi^4} & n = m \text{ and } n \neq 0 \\ \frac{9}{32 n^4 \pi^4} & n = -m \text{ and } n \neq 0 \\ \frac{2\,n^2 - 3\,n\,m + 2\,m^2}{8\,n^2\,m^2\,(n-m)^2\,\pi^4} & |n| \neq |m| \text{ and } n \neq 0 \neq m \end{cases} \tag{5.16}$$

up to terms of order J^4.

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5.3 $W_1(x,y)$ to next to leading order

Inserting the genus expansion (3.7) into (5.3) we can easily determine the genus one contribution to W_1. From (5.3) and (5.10) we get

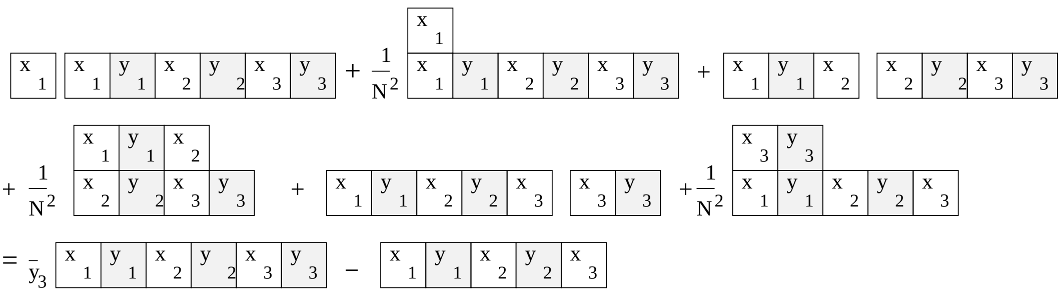
$$W_1^{(1)}(x,y) = \frac{x}{x\bar{y}-1} U_1^{(0)}(x;x,y) = \frac{x^2\bar{y}^2}{(x\bar{y}-1)^5}\,. \tag{5.17}$$

6. Functionals needed for the one-loop, genus two computation

In this section we shall determine the third order contribution to (3.3), the next to leading order contributions to the functionals (3.4) and (3.5) as well as the leading order contribution to the functions $U_{1,1}$, $W_{1,2}$, U_2 and W_3. We shall start by the latter ones and work our way toward the first ones.

6.1 W_3, U_2, $W_{2,1}$ and $U_{1,1}$ to leading order

Considering the field redefinition $\delta Z = (1/(x_1 - Z))(1/(\bar{y}_1 - \bar{Z}))(1/(x_2 - Z))(1/(\bar{y}_2 - \bar{Z}))\times(1/(x_3 - Z))(1/(\bar{y}_3 - \bar{Z}))$ we obtain

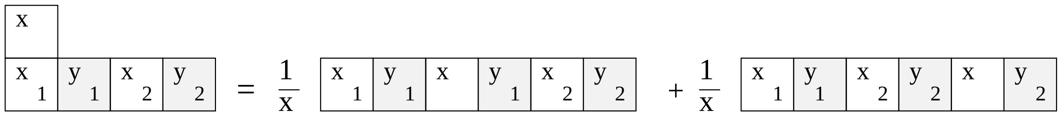


$$\begin{aligned}
&W_3(x_1,y_1,x_2,y_2,x_3,y_3)\left(\bar{y}_3 - \frac{1}{x_1}\right) = \\
&\quad = (1+W_1(x_3,y_3))\frac{W_2(x_1,y_1,x_2,y_2)-W_2(x_3,y_1,x_2,y_2)}{x_3-x_1} + \\
&\qquad + W_2(x_2,y_2,x_3,y_3)\frac{W_1(x_1,y_1)-W_1(x_2,y_1)}{x_2-x_1} + \\
&\qquad + \frac{1}{N^2}\frac{W_{1,2}(x_1,y_1;x_2,y_2,x_3,y_3)-W_{1,2}(x_2,y_1;x_2,y_2,x_3,y_3)}{x_2-x_1} + \\
&\qquad + \frac{1}{N^2}\frac{W_{1,2}(x_3,y_3;x_1,y_1,x_2,y_2)-W_{1,2}(x_3,y_3;x_3,y_1,x_2,y_2)}{x_3-x_1} + \\
&\qquad + \frac{1}{N^2}U_3(x_1;x_1,y_1,x_2,y_2,x_3,y_3)\,.
\end{aligned} \tag{6.1}$$

From here we can immediately get the genus zero contribution to W_3 from our knowledge of the genus zero contribution to W_1 and W_2. The result reads

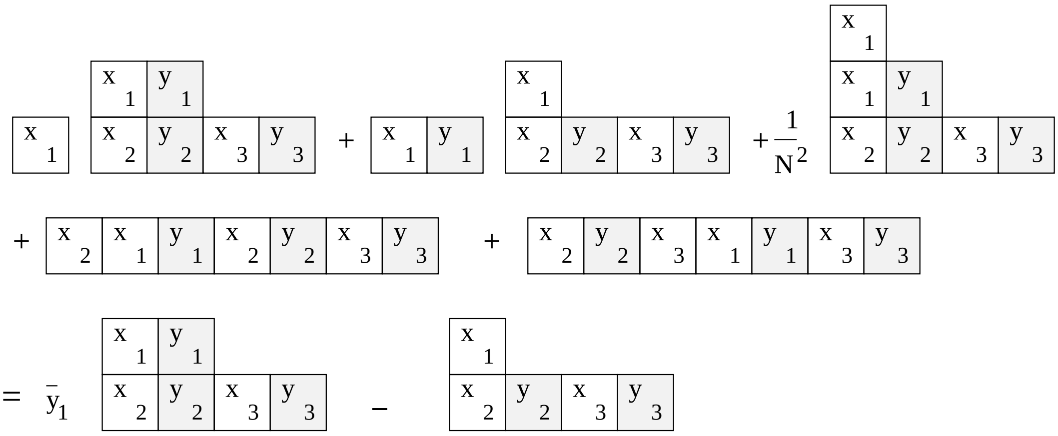
$$W_3^{(0)}(x_1,y_1,x_2,y_2,x_3,y_3) = \frac{\prod_i x_i\bar{y}_i}{\prod_{i,j}(x_i\bar{y}_j-1)} \det\begin{pmatrix} 1 & x_1\bar{y}_2 & 1 \\ 1 & 1 & x_2\bar{y}_3 \\ x_3\bar{y}_1 & 1 & 1 \end{pmatrix}. \tag{6.2}$$

Next, performing the change of variables $\delta\bar{Z} = (1/(x-Z))\,\mathrm{tr}(1/(x_1-Z)(\bar{y}_1-\bar{Z})(x_2-Z)(\bar{y}_2-\bar{Z}))$ leads to the following simple relation



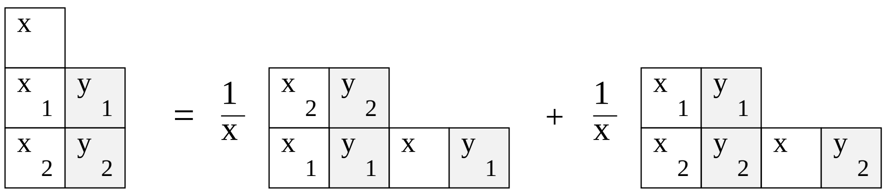
$$U_2(x;x_1,y_1,x_2,y_2) = \frac{1}{x}\left(W_3(x_1,y_1,x,y_1,x_2,y_2) + W_3(x_1,y_1,x_2,y_2,x,y_2)\right) \tag{6.3}$$

and from which we easily get $U_2^{(0)}$ by inserting (6.2). Furthermore, choosing the field redefinition $\delta Z = (1/(x_1-Z))(1/(\bar{y}_1-\bar{Z}))\,\mathrm{tr}(1/(x_2-Z)(\bar{y}_2-\bar{Z})(x_3-Z)(\bar{y}_3-\bar{Z}))$ we find



$$\begin{aligned}
W_{1,2}(x_1,y_1;x_2,y_2,x_3,y_3)&\left(\bar{y}_1 - \frac{1}{x_1}\right) = \\
&= (1+W_1(x_1,y_1))U_2(x_1;x_2,y_2,x_3,y_3) + \\
&\quad + \frac{W_3(x_2,y_1,x_2,y_2,x_3,y_3) - W_3(x_1,y_1,x_2,y_2,x_3,y_3)}{x_1-x_2} + \\
&\quad + \frac{W_3(x_3,y_3,x_2,y_2,x_3,y_1) - W_3(x_3,y_3,x_2,y_2,x_1,y_1)}{x_1-x_3} + \\
&\quad + \frac{1}{N^2}U_{1,2}(x_1;x_1,y_1;x_2,y_2,x_3,y_3)\,,
\end{aligned} \tag{6.4}$$

where $W_{1,2}$ is now expressed in terms of already known quantities. Furthermore, making use of the change of variable $\delta\bar{Z} = (1/(x-Z))\,\mathrm{tr}(1/(x_1-Z)(\bar{y}_1-\bar{Z}))\,\mathrm{tr}(1/(x_2-Z)(\bar{y}_2-\bar{Z}))$ we obtain the following simple relation



$$U_{1,1}(x;x_1,y_1;x_2,y_2) = \frac{1}{x}\left(W_{1,2}(x_2,y_2;x_1,y_1,x,y_1) + W_{1,2}(x_1,y_1;x_2,y_2,x,y_2)\right). \tag{6.5}$$

6.2 W_2, U_1 and $W_{1,1}$ to next to leading order

Inserting the genus expansion (3.7) and (3.8) into the relevant loop equations (5.6), (5.9) and (5.12) we can easily determine the genus one contribution to U_1, W_2 and $W_{1,1}$. From (5.6) we find (using MATHEMATICA)

$$\begin{aligned}
W_2^{(1)}(x_1,y_1,x_2,y_2) &= \frac{x_1}{x_1\bar{y}_2-1}\Bigg\{\left(1+W_1^{(0)}(x_2,y_2)\right)\frac{W_1^{(1)}(x_1,y_1)-W_1^{(1)}(x_2,y_1)}{x_2-x_1}+\\
&\quad +\frac{W_1^{(0)}(x_1,y_1)-W_1^{(0)}(x_2,y_1)}{x_2-x_1}W_1^{(1)}(x_2,y_2)+\\
&\quad +\frac{W_{1,1}^{(0)}(x_1,y_1;x_2,y_2)-W_{1,1}^{(0)}(x_2,y_1;x_2,y_2)}{x_2-x_1}+\\
&\quad +U_2^{(0)}(x_1;x_1,y_1,x_2,y_2)\Bigg\}\\
&= \frac{\prod_i x_i\bar{y}_i}{\prod_{i,j}(x_i\bar{y}_j-1)^5}\mathrm{Pol}_{28}(x_1,\bar{y}_1,x_2,\bar{y}_2)\,,
\end{aligned} \tag{6.6}$$

where $\mathrm{Pol}_d(x_1,\bar{y}_1,x_2,\bar{y}_2)$ denotes a polynomial of degree d in x_1, $\bar{y}_1$, x_2, $\bar{y}_2$. If we carry out the discrete Fourier transform and take the large-J limit we get

$$\begin{aligned}
D_{n,m}^{(1)} &= D_{m,n}^{(1)} = \sum_{p,q=0}^{J}\frac{1}{N}\left\langle \mathrm{tr}\, Z^p\bar{Z}^qZ^{J-p}\bar{Z}^{J-q}\right\rangle_{h=1} e^{(2\pi i/J)(np-mq)}\\
&= \oint\frac{dr}{2\pi i}r^{J+1}W_2^{(1)}\left(\sqrt{r}e^{-i\pi n/J},\sqrt{r}e^{-i\pi m/J};\sqrt{r}e^{i\pi n/J},\sqrt{r}e^{i\pi m/J}\right)e^{i\pi(n-m)}\\
&= J^7\begin{cases}\frac{1}{240} & n=m=0\\ \frac{6-\pi^2n^2}{48\,n^4\pi^4} & n\neq 0, m=0\\ \frac{315-120\pi^2n^2+32\pi^4n^4}{7680\,n^6\pi^6} & |n|=|m| \text{ and } n\neq 0\\ \frac{3(n^6+m^6)+4\pi^2(n^6m^2+m^6n^2)-15(m^4n^2+n^4m^2)-8m^4n^4\pi^2}{48\,n^2\,m^2\,(m-n)^4\,(m+n)^4\pi^6} & |n|\neq|m| \text{ and } n\neq 0\neq m\end{cases}
\end{aligned} \tag{6.7}$$

up to terms of order J^6.

Furthermore, from (5.9)

$$\begin{aligned}
U_1^{(1)}(x;x_1,y_1) &= \frac{1}{x}W_2^{(1)}(x,y_1,x_1,y_1)\\
&= \frac{x_1\bar{y}_1^2}{(1-x\bar{y}_1)^6(1-x_1\bar{y}_1)^6}\Big\{1+4\bar{y}_1(x_1+x)-34\bar{y}_1^2xx_1+18\bar{y}_1^3(xx_1(x+x_1))+\\
&\quad +\bar{y}_1^4xx_1(-10x^2+21xx_1-10x_1^2)+\\
&\quad +2\bar{y}_1^5xx_1(x^3-6x^2x_1-6xx_1^2+x_1^3)+\\
&\quad +\bar{y}_1^6x^2x_1^2(3x^2+2xx_1+3x_1^2)\Big\}\,.
\end{aligned} \tag{6.8}$$

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Finally, from (5.12) one gets

$$
\begin{aligned}
W_{1;1}^{(1)}&(x_1,y_1;x_2,y_2) = \\
&= \frac{x_1}{x_1\bar{y}_1-1}\Bigg\{(1+W_1^{(0)}(x_1,y_1))U_1^{(1)}(x_1;x_2,y_2)+W_1^{(1)}(x_1,y_1)U_1^{(0)}(x_1;x_2,y_2)+ \\
&\qquad\qquad + \frac{W_2^{(1)}(x_1,y_1,x_2,y_2)-W_2^{(1)}(x_2,y_1,x_2,y_2)}{x_2-x_1}+U_{1,1}^{(0)}(x_1;x_1,y_1;x_2,y_2)\Bigg\} \\
&= \frac{\prod_i x_i\bar{y}_i}{\prod_{i,j}(x_i\bar{y}_j-1)^6}\mathrm{Pol}_{36}(x_1,\bar{y}_1,x_2,\bar{y}_2)\,. \qquad (6.9)
\end{aligned}
$$

In this case carrying out the discrete Fourier transform and taking the large-J limit gives

$$
C_{n,m}^{(1)} = C_{m,n}^{(1)} = J^9 \begin{cases} \frac{73}{181440} & n=m=0 \\ \frac{7}{16n^8\pi^8}-\frac{13}{96n^6\pi^6}+\frac{7}{320n^4\pi^4}-\frac{1}{480n^2\pi^2} & n\neq 0 \text{ and } m=0 \\ \frac{143}{2048n^8\pi^8}-\frac{107}{768n^6\pi^6}+\frac{3}{160n^4\pi^4}-\frac{1}{2016n^2\pi^2} & n=m \text{ and } n\neq 0 \\ \frac{7}{4096n^8\pi^8}-\frac{245}{3072n^6\pi^6}+\frac{13}{1280n^4\pi^4} & n=-m \text{ and } n\neq 0 \\ \frac{1}{192m^4n^4(m-n)^6(m+n)^4\pi^8}\times\mathrm{Pol}_{14}(m,n) & |n|\neq|m| \text{ and } n\neq 0\neq m\,, \end{cases}
$$

where

$$
\begin{aligned}
\mathrm{Pol}_{14}(m,n) \;=\; & 3\Big(2\,m^{10}-m^9\,n-13\,m^8\,n^2+9\,m^7\,n^3+63\,m^6\,n^4+120\,m^5\,n^5+ \\
& \quad + 63\,m^4\,n^6+9\,m^3\,n^7-13\,m^2\,n^8-m\,n^9+2\,n^{10}\Big) - \\
& - \left(m^2-n^2\right)^2\Big(12\,m^8-18\,m^7\,n-20\,m^6\,n^2+37\,m^5\,n^3+ \\
& \quad + 38\,m^4\,n^4+37\,m^3\,n^5-20\,m^2\,n^6-18\,m\,n^7+12\,n^8\Big)\,\pi^2+ \\
& + m^2\,n^2\left(m^2-n^2\right)^4\left(2\,m^2-3\,m\,n+2\,n^2\right)\,\pi^4 \qquad (6.10)
\end{aligned}
$$

and where we have neglected terms of order J^8. Notice that the expressions above constitute the contribution to the mixing matrix coming from connected diagrams only. If one includes also disconnected ones one reproduces the expressions given in [9].[2]

6.3 $W_1(x,y)$ to third order

Making use of (5.3) and (6.8) we get

$$
W_1^{(2)}(x,y) = \frac{x}{x\bar{y}-1}U_1^{(1)}(x;x,y) = \frac{x^2y^2(1+12x\bar{y}+8x^2\bar{y}^2)}{(x\bar{y}-1)^9}\,. \qquad (6.11)
$$

[2] A detailed comparison is not possible in the general $|n|\neq|m|$ case and there seems to be a discrepancy in the two last terms of the $n=-m$ case.

7. The general case

From the examples above it should be clear how to choose the appropriate field redefinitions needed for the derivation of the loop equation associated with a given generating functional. Here we shall write down the most general loop equations and show how they allow us to determine recursively any multi-loop correlator, i.e. any expectation value of traces of words to any order in the genus expansion.

7.1 One-trace functions

Considering the change of variable $\delta\bar{Z} = (1/(x_1 - Z))(1/(\bar{y}_1 - \bar{Z}) \cdots (1/(x_l - Z))(1/(\bar{y}_l - \bar{Z}))$ with $l \geq 2$ we obtain the following relation

$$\begin{aligned}\left(\frac{1}{x_1} - \bar{y}_l\right) W_l(x_1,\ldots,y_l) = \sum_{k=2}^{l} \frac{W_{k-1}(x_1,y_1,\ldots,y_{k-1}) - W_{k-1}(x_k,y_1,\ldots,y_{k-1})}{x_1 - x_k} \times \\ \times \left(\delta_{k,l} + W_{l-k+1}(x_k,y_k,\ldots,x_l,y_l)\right) + O\left(\frac{1}{N^2}\right). \qquad (7.1)\end{aligned}$$

Here W_l is expressed entirely in terms of W_k with $k \leq l-1$. Having determined the planar contribution to $W_1(x,y)$ (cf. eqn. (5.4)) we can by means of (7.1) determine recursively the planar contribution to any one-trace function W_l. From the structure of the recursion relation and the explicit expression for $W_1^{(0)}(x,y)$ it follows that planar one-trace functions only have singularities in the form of single poles. More precisely we have

$$W_l^{(0)}(x_1,y_1,\ldots,x_l,y_l) = \frac{\prod_i x_i\bar{y}_i}{\prod_{i,j}(x_i\bar{y}_j - 1)} \mathrm{Pol}(x_1,\bar{y}_1,\ldots,x_l,\bar{y}_l)\,, \qquad (7.2)$$

where $\mathrm{Pol}(x_1,\bar{y}_1,\ldots,x_l,\bar{y}_l)$ is a polynomial of degree $l-2$ in each of its variables. In the case $l=3$ (and trivially in the case $l=2$) this polynomial could be expressed as a determinant (cf. equations (5.7) and (6.2)) but it does not seem that a similar simplification occurs for higher values of l. It would be most interesting, though, to find a closed expression for W_l for general l.

7.2 Multi-trace functions

In the case of the W-functions it is convenient to consider separately the cases $l_1 = 1$ and $l_1 > 1$. For $l_1 = 1$ and ($n \geq 2$) we have

$$\begin{aligned}&\left(\bar{y}_{1,1} - \frac{1}{x_{1,1}}\right) W_{1,l_2,\ldots,l_n}(x_{1,1},y_{1,1};x_{2,1},\ldots,y_{n,l_n}) = \\ &\quad = (1 + W(x_{1,1},y_{1,1}))U_{l_2,\ldots,l_n}(x_{1,1};x_{2,1},\ldots,y_{n,l_n}) + \\ &\quad + \sum_{k=2}^{n}\sum_{j=1}^{l_k} \frac{1}{x_{k,j} - x_{1,1}} \Big\{ W_{l_2,\ldots,l_k+1,\ldots,l_n}(\ldots,y_{k,j-1},x_{1,1},y_{1,1},x_{k,j},y_{k,j},\ldots) - \\ &\qquad\qquad - W_{l_1,\ldots,l_k+1,\ldots,l_n}(\ldots,y_{k,j-1},x_{k,j},y_{1,1},x_{k,j},y_{k,j},\ldots)\Big\} + \\ &\quad + \frac{1}{N^2} U_{1,l_2,\ldots,l_n}(x_{1,1};x_{1,1},y_{1,1};x_{2,1},\ldots,y_{n,l_n}) \qquad (7.3)\end{aligned}$$

whereas for $l_1 > 1$ the relevant loop equation reads

$$
\begin{aligned}
&\left(\bar{y}_{1,l_1} - \frac{1}{x_{1,1}}\right) W_{l_1,\dots,l_n}(x_{1,1},\dots,y_{n,l_n}) = \\
&= \sum_{j=2}^{l_1} W_{l_1+1-j,\dots,l_n}(x_{1,j},\dots,y_{n,l_n})(1-\delta_{n,1}) \times \\
&\quad \times \left\{ \frac{W_{j-1}(x_{1,1},y_{1,1},\dots,y_{1,j-1}) - W_{j-1}(x_{1,j},y_{1,1},\dots,y_{1,j-1})}{x_{1,j}-x_{1,1}} \right\} + \\
&\quad + \sum_{j=2}^{l_1} (W_{l_1+1-j}(x_{1,j},\dots,y_{1,l_1}) + \delta_{j,l_1}) \times \\
&\quad \times \left\{ \frac{W_{j-1,\dots,l_n}(x_{1,1},\dots,y_{1,j-1};x_{2,1},\dots,y_{n,l_n})}{x_{1,j}-x_{1,1}} - \right. \\
&\qquad \left. - \frac{W_{j-1,\dots,l_n}(x_{1,j},\dots,y_{1,j-1};x_{2,1},\dots,y_{n,l_n})}{x_{1,j}-x_{1,1}} \right\} + \\
&\quad + \sum_{k=2}^{n}\sum_{j=1}^{l_k} \frac{1}{x_{k,j}-x_{1,1}} \Big\{ W_{l_2,\dots,l_k+l_1,\dots,l_n}(x_{2,1},\dots,y_{k,j-1},x_{1,1},\dots,y_{1,l_1},x_{k,j},\dots,y_{n,l_n}) - \\
&\qquad\qquad - W_{l_1,\dots,l_k+l_1,\dots,l_n}(x_{2,1},\dots,y_{k,j-1},x_{k,j},\dots,y_{1,l_1},x_{k,j},\dots,y_{n,l_n}) \Big\} + \\
&\quad + \frac{1}{N^2} \sum_{j=2}^{l_1} \frac{1}{x_{1,j}-x_{1,1}} \Big\{ W_{j-1,l_1+1-j,l_2,\dots,l_n}(x_{1,1},y_{1,1},\dots,y_{1,j};x_{1,j},\dots,y_{n,l_n}) - \\
&\qquad\qquad - W_{j-1,l_1+1-j,\dots,l_n}(x_{1,j},y_{1,1},\dots,y_{1,j};x_{1,j},\dots,y_{n,l_n}) \Big\} + \\
&\quad + \frac{1}{N^2} U_{l_1\dots l_n}(x_{1,1};x_{1,1},\dots y_{n,l_n}) \,.
\end{aligned}
\tag{7.4}
$$

These equations are to be supplemented by the loop equations for the U-functions which take the simpler form

$$
\begin{aligned}
&x\, U_{l_1,\dots,l_n}(x;x_{1,1},\dots,y_{n,l_n}) = \\
&\quad = \sum_{k=1}^{n}\sum_{j=1}^{l_k} W_{l_1,\dots,l_k+1,\dots,l_n}(x_{1,1},\dots;x_{k,1},\dots,y_{k,j},x,y_{k,j},x_{k,j+1},\dots,y_{k,l_k};\dots,y_{n,l_n}) \,.
\end{aligned}
\tag{7.5}
$$

The relations (7.3), (7.4) and (7.5) constitute a triangular set of equations which allows us to determine any multi-loop function to any order in the genus expansion. For a finite number of loops and finite genus only a finite number of operations are needed. Below we shall make this statement more precise.

First, let us introduce an ordering of the Young diagrams representing W-functions. A Young diagrams $Y_{k_1,\dots,k_m}$ is said to be smaller than a Young diagram $Y_{l_1,\dots,l_n}$ (representing loop-functions $W_{k_1,\dots,k_m}$ and $W_{l_1,\dots,l_n}$, respectively) if:

$$
m + \sum_{i=1}^{m} k_i < n + \sum_{i=1}^{n} l_i \,. \tag{7.6}
$$

Next, let us consider the loop equations (7.3) and (7.4) (using (7.5)): it is clear that all the leading order diagrams on the RHS are smaller than the diagram on the LHS. This means that at the planar level a W-function corresponding to a certain Young diagram can be expressed entirely in terms of planar W-functions corresponding to smaller Young diagrams. We thus get a closed equation for any genus zero W-function and clearly also for any genus zero U-function (cf. equation (7.5)).

Proceeding to higher genera, we have in our loop equations (7.3) and (7.4) two types of terms which carry a factor $1/N^2$; U-terms and W-terms. Compared to the object we are interested in, the W-terms correspond to Young diagrams where one extra line has been added while the number of boxes has been kept fixed. The U-terms, on the other hand, correspond via (7.5) to Young diagrams where two extra boxes have been added while the number of lines has been kept fixed. This means that the genus g contribution to a W-function described by a Young diagram with $2K$ boxes and n lines can be expressed entirely in terms of genus zero W-functions corresponding to Young diagrams having at most $2K+2g$ boxes and $n+g-1$ lines. Clearly, we have a triangular set of equations which allows us to determine any expectation value of traces of words to any order in the genus expansion.

8. Exactly calculable correlators

As shown in reference [14, 25] it is possible to find exact expressions for the expectation values encoded in the following generating functionals

$$H_n(x_1,\ldots,x_n,y) = N^{n-1}\left\langle \mathrm{tr}\,\frac{1}{x_1-Z}\ldots\mathrm{tr}\,\frac{1}{x_n-Z}\,\mathrm{tr}\,\frac{1}{\bar{y}-\bar{Z}}\right\rangle_{\mathrm{conn}} \tag{8.1}$$

as well as [14]

$$G(x_1,x_2,y_1,y_2) = N^2\left\langle \mathrm{tr}\,\frac{1}{x_1-Z}\,\mathrm{tr}\,\frac{1}{x_2-Z}\,\mathrm{tr}\,\frac{1}{\bar{y}_1-\bar{Z}}\,\mathrm{tr}\,\frac{1}{\bar{y}_2-\bar{Z}}\right\rangle_{\mathrm{conn}}. \tag{8.2}$$

From the generating functionals (8.1) and (8.2) it is possible using again loop equations to derive exact expressions for yet other generating functionals. The functions $W_1(x,y)$ and $U_1(x';x,y)$ can be determined in full generality whereas the method only gives the remaining W- and U-functions in certain limits where typically a number of their arguments are sent to ∞. To obtain the all genus version of $W_1(x,y)$ one considers the field redefinition $\delta\bar{Z} = (1/(x-Z))\,\mathrm{tr}\,(1/(\bar{y}-\bar{Z}))$ which leads to the following first order differential equation

$$-\partial_{\bar{y}} W_1(x,y) = x H_1(x,y) \tag{8.3}$$

which is to be supplemented by the boundary condition

$$W_1(x,y) \to \frac{1}{x\bar{y}}, \qquad \text{as } |x|,|y|\to\infty\,. \tag{8.4}$$

The equation (8.3) is of course nothing but the generating functional version of the simple relation

$$\left\langle \mathrm{tr}\,Z^J\,\mathrm{tr}\,\bar{Z}^J\right\rangle = J\left\langle \mathrm{tr}\,Z^{J-1}\bar{Z}^{J-1}\right\rangle \tag{8.5}$$

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which implies

$$W_1(x,y) = \sum_{J=0}^{\infty} \frac{1}{N^J x^{J+1} \bar{y}^{J+1}} \frac{1}{(J+1)(J+2)} \left\{ \frac{(N+J+1)!}{(N-1)!} - \frac{N!}{(N-J-2)!} \right\} \tag{8.6}$$

which can also be written

$$W_1(x,y) = \sum_{k=0}^{\infty} \frac{1}{N^{2k}} f_{2k+1}(x\bar{y}) \tag{8.7}$$

with $f_1(x) = 1/(x-1)$ and:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_{k+1} = \frac{1}{1-x} \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}^2}{\mathrm{d}x^2} f_k \tag{8.8}$$

and $f_k(x) = O(1/x^k)$ for large x. The coefficient of $1/(x-1)^{2k-1}$ as $x \to 1$ is $(2k-3)!!/k$.

In the case of $U_1(x';x,y)$ one chooses the field redefinition $\delta\bar{Z} = (1/(x-Z)) \times \times \operatorname{tr}(1/(x'-Z)) \operatorname{tr}(1/(\bar{y}-\bar{Z}))$ and obtains

$$-\partial_{\bar{y}} U_1(x';x,y) = x H_2(x',x,y) \tag{8.9}$$

and the appropriate boundary condition in this case reads

$$U_1(x';x,y) \to \frac{1}{(x'\bar{y})^2 x} \qquad \text{as } |x'|, |x|, |y| \to \infty\,. \tag{8.10}$$

Expressed in terms of expectation values (8.9) reads

$$\left\langle \operatorname{tr} Z^J \operatorname{tr} Z^K \operatorname{tr} \bar{Z}^{J+K} \right\rangle = (J+K) \left\langle \operatorname{tr} Z^K \operatorname{tr} \bar{Z}^{J+K-1} Z^{J-1} \right\rangle \tag{8.11}$$

which has the obvious generalization with $J = \sum_{i=1}^{k} J_i$

$$\left\langle \operatorname{tr} \bar{Z}^J \prod_{i=1}^{k} \operatorname{tr} Z^{J_i} \right\rangle = J \left\langle \operatorname{tr} \bar{Z}^{J-1} Z^{J_1-1} \prod_{i=2}^{k} \operatorname{tr} Z^{J_i} \right\rangle\,. \tag{8.12}$$

9. Conclusion

With this work we have added pp-wave physics and $\mathcal{N}=4$ SYM to the long list of areas where classical matrix model techniques have proven very efficient.

As explained in section 2, evaluating a typical correlation function in the BMN sector of $\mathcal{N}=4$ SYM can be reduced to evaluating the expectation value of a product of traces of words in a zero-dimensional gaussian complex matrix model and subsequently carrying out a discrete Fourier transformation. Simple correlators can be obtained by purely combinatorial arguments but such arguments become more and more involved (and correspondingly less and less reliable) the more words enter the correlators and the higher genus one is aiming at. With our loop equation based technique, however, we can by analytical manipulations reach any multi-word correlator to any order in the genus expansion. Furthermore, by working with generating functionals we trade the process of Fourier transformation for simple contour integration.

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There are several directions of investigation where our technique would be most useful. One is the investigation of operators with more impurities than the traditionally studied BMN operators of equation (2.7). Such operators would correspond to string states with many oscillators excited and determining their correlators would imply evaluating expectation values of many letter words. Such words are encoded in U- and W-functions whose sub-scripts are large and these are of course accessible with our method.

As mentioned several times our method would also allow us to calculate the higher genera, one-loop corrections to the anomalous dimension of the BMN operators (2.7). In fact, we have already evaluated all expectation values needed for the genus two calculation. As pointed out in [14], completing this calculation would allow one to check whether the effective string coupling constant in the pp-wave/BMN correspondence is indeed $g_2\sqrt{\lambda'}$ as suggested in [9, 13]. Finally, it is possible that like in the one-loop case two- and higher loop computations on the gauge theory side can be reduced to pure matrix model computations and then obviously our method will again be in demand.

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